IMPACT OF LIQUIDS OF DIFFERENT DENSITIES

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A two-dimensional self-similar flow generated, in particular, by plunging breaking water waves is considered. The interface between the liquids is considered as a solid surface which shape is determined from the condition that the pressure distribution on both sides of the interface is the same. An integral hodograph method is applied to solve the boundary-value problems for each side of the interface. The problem is reduced to a system of integro-differential equations that are solved numerically using the method of successive approximations.

1. INTRODUCTION

The collisions between liquids, or between a liquid and a liquid-like solid such as granular materials, at extremely high speeds is a commonly observed phenomenon in nature and various engineering applications. Examples include plunging breaking water waves, liquid drops impacting a free surface of the same or another liquid, debris, snow (avalanche), lava (volcano) entering liquid surface. Investigations and reviews of these phenomena were presented by Yarin [1], Thoroddsen, Etoh & Takehara [2], Kiger & Duncan [3], Tran et al. [4] and many other authors.

In the present study we consider the impact problem of liquids of different densities through two wedges, based on the velocity potential theory for the flow within each incompressible fluid. As observed in the case of impact of liquid with the same density (Howison, Ockendon, Oliver, Purvis & Smith [5], Semenov, Wu & Oliver [6]), splash jets are usually formed by the impact. The speed of the splash jet depends on the shapes of the colliding liquids. It may exceed their initial relative speed and cause secondary impacts. This presents one of the major obstacles to obtaining solution, together with the nonlinear boundary conditions on the unknown free surface shape. For collision of liquids with different densities ρ', ρ considered here, the problem is further complicated by the unknown interface which separates the two fluids. The determination of the shape of the interface is based on the conditions that the pressure and the normal velocity of the flow are continuous across the interface, while the tangential velocity may be discontinuous. The developed methodology is also relevant for impact between a liquid wedge and the perforated or porous solid surface similar to that presented by Iafrati, Miloh & Korobkin [7] and Molin & Korobkin [8].

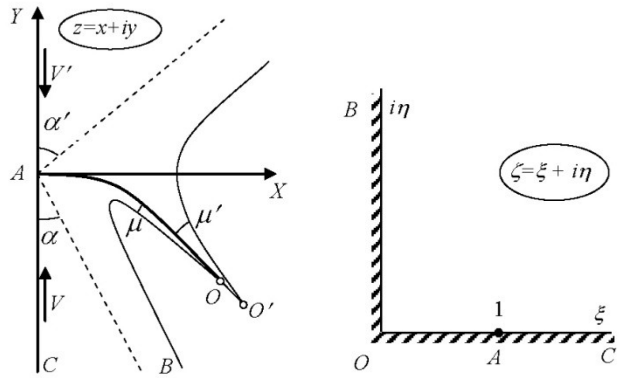
The integral hodograph method [9, 10] is employed to derive analytical expressions for the complex velocity potential, the conjugate of complex velocity, and the mapping function. The problem is reduced to a system of integro-differential equations in terms of the velocity magnitude and the angle of the velocity direction relative to the liquid boundary and the interface, respectively. The solution of these equations is obtained by the numerical discretisation and iteration. The solution procedure has been validated by comparing the obtained results for the same density with those obtained from a different formulation of the boundary-value problem [6]. Various results are presented in the form of streamline patterns, the pressure and velocity distributions along the interface, angle of the splash jet.

The asymptotic analysis has also been made for two liquid wedges with large density ratio, or $\rho'/\rho >> 1$ which corresponds to the problem of a wedge-shaped avalanche entering water at high speed.

2. NONLINEAR ANALYSIS

A sketch of the problem and the definitions of the geometric parameters are shown in Fig. 1*a*. The liquid wedges are assumed to be symmetric about the $y-$axis. Their half-angles are α and α' respectively, densities ρ and ρ' respectively. They move in opposite directions with V and V' respectively, and their tips meet at point A at time $t=0$, where the origin of the Cartesian coordinate system is chosen. The impact problem is solved in each liquid wedge on each side of the interface OA where the same normal velocity is used. The pressure continuity across OA is used to determine its shape.

For a constant impact velocity of the liquid wedge, the time-dependent problem in the physical plane $Z = X + iY$ can be written in the stationary plane $z = x + iy$ in terms of the self-similar variables $x = X/(Vt)$, $y = Y/(Vt)$, $s = S/(Vt)$ where V is the velocity magnitude at infinity BC in the physical plane and S is the spatial length coordinate along the free surface.



(*a*) (*b*)

Figure 1. Sketch of the collision of two liquid wedges: (*a*) the physical plane; (*b*) the parameter plane corresponding to the lower liquid wedge.

The complex velocity potential $W(Z,t) = \Phi(X,Y,t) + i\Psi(X,Y,t)$ for self-similar flows is written in the form

$$W(Z,t) = V^2 t w(z) \tag{1}$$

The problem is to determine the function $w(z)$ which conformally maps the stationary plane z onto the complex velocity potential region w. As shown in Fig.1*b* for the lower wedge while similar procedure can be used the upper wedge, we choose the first quadrant of the $\zeta-$ plane as the parameter region to derive expressions for the conjugate of the nondimensional complex velocity, dw/dz, and for the derivative of the complex potential, $dw/d\zeta$, both as functions of the variable ζ. Once these functions are found, the velocity field and the mapping function $z = z(\zeta)$ can be determined by integration of the ratio ($dw/d\varsigma$)/(dw/dz).

Conformal mapping allows us to fix three arbitrary points in the parameter region, which are O (tip of the interface), BC (at infinity), and A (the stagnation point), as shown in figure 1*b*. In this plane, the positive part of the imaginary axis ($0<\eta<\infty, \xi=0$) corresponds to the free surface OB. The interval ($0<\xi<1$, $\eta=0$) of the real axis corresponds to the interface OA, and the rest of the positive real axis ($1<\xi<\infty$, $\eta=0$) corresponds to the symmetry line AC. The point $\zeta=1$ is the image of the stagnation point A in the stationary plane z.

The boundary-value problems for the complex velocity function, dw/dz, and for the derivative of the complex potential, $dw/d\zeta$, can be formulated in the parameter plane. Then, applying the integral formulae determining an analytical function from its modulus and argument, and from its argument on the boundary of the first quadrant [10], respectively, we obtain the following expression for the complex velocity and for the derivative of the complex potential.

$$\frac{dw}{dz} = v_0^{-i\beta_0}\sqrt{\frac{\zeta-1}{\zeta+1}}\exp\left[\begin{array}{l}\dfrac{1}{\pi}\displaystyle\int_0^1 \frac{d\beta}{d\xi}\ln\left(\frac{\xi-\zeta}{\xi+\zeta}\right)d\xi \\ -\dfrac{i}{\pi}\displaystyle\int_0^\infty \frac{d\ln v}{d\eta}\ln\left(\frac{\zeta-i\eta}{\zeta+i\eta}\right)d\eta\end{array}\right], \tag{2}$$

$$\frac{dw}{d\zeta} = K\zeta^{-1-2\alpha/\pi}\exp\left[\begin{array}{l}\dfrac{1}{\pi}\displaystyle\int_0^1 \frac{d\gamma}{d\xi}\ln(\xi^2-\zeta^2)d\xi \\ +\dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{d\theta}{d\eta}\ln(\zeta^2+\eta^2)d\eta\end{array}\right], \tag{3}$$

where K is a real scale factor, $v_0 = v(\eta)_{\eta=0}$ is the velocity magnitude at point O, $\theta(\eta) = \tan^{-1}(v_n/v_s)$ is the angle between the velocity vector and the free surface, and $\gamma(\xi) = \tan^{-1}(v_n/v_s)$ is the angle between the velocity vector and the interface. The normal component of the velocity along the interface is $v_n = \mathrm{Im}(\bar{z}e^{i\delta})$, where $\bar{z}$ is complex conjugate coordinate and $\tan\delta$ is the slope to the interface, the tangential component of the velocity, v_s, is determined from Eq. (2) and $\beta(\xi) = \delta[s(\xi)] - \gamma(\xi)$.

At time $t=0$, the tip of the interface, point O, and the stagnation point A coincide. The tip of the interface moves with the velocity of the liquid at point O having the magnitude v_0 and the angle β_0 with the $x-$ axis. Thus, the length of the interface OA is determining from the equation $|z_O| = \left|\int_0^{s^*} e^{i\delta(s)}ds\right| = v_0$. The functions $v(\eta)$ and $\theta(\eta)$ are determined from dynamic and kinematic boundary conditions which for an arbitrary self-similar flow take the form [9]:

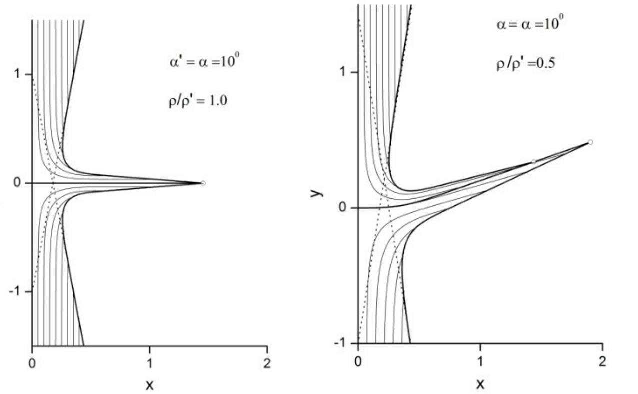
$$\frac{d\theta}{d\eta} = \frac{v + s\cos\theta}{s\sin\theta}\frac{d\ln v}{d\eta}, \tag{4}$$

$$\frac{1}{\tan\theta}\frac{d\ln v}{d\eta} = \frac{d}{d\eta}\left[\arg\left(\frac{dw}{dz}\right)\right]. \tag{5}$$

By choosing in the Bernoulli equation with the reference point at the stagnation point A, putting there $S=0$ and $W(Z_A,t)=0$, and taking advantage of the selfsimilarity of the flow, we can determine the pressure at any point of the flow region

$$c_p^* = \frac{2(P-P_A)}{\rho V_O^2} = \mathrm{Re}\left(-2w + 2z\frac{dw}{dz}\right) - \left|\frac{dw}{dz}\right|^2 . \qquad (6)$$

The method of successive approximations is used to determine the interface. At each iteration the system of integro-differential equations determining the functions $v(\eta)$, $\theta(\eta)$ and $\gamma(\xi)$ is solved using the iteration procedure. Solving the same problem for the upper liquid wedge, the pressure distribution along the upper side of the interface, $c_p^{*\prime}$, is obtained. The iteration continues until the both lower and upper side pressure distributions becomes the same.



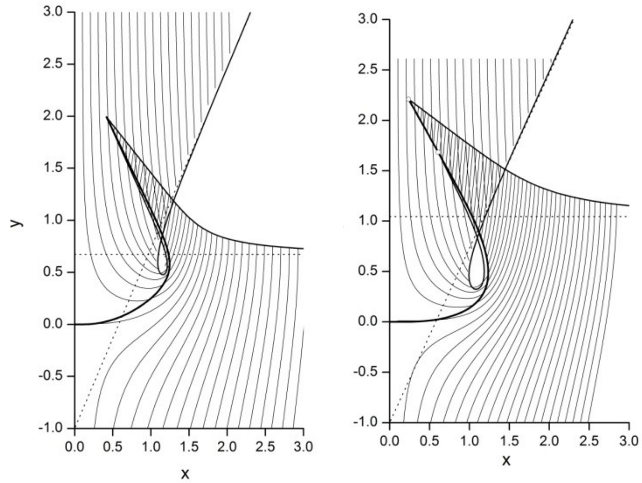
(a) (b)

Figure 2. Streamlines for $\alpha' = \alpha = 10^\circ$ at density ratio (a) $\rho/\rho' = 1$ and (b) $\rho/\rho' = 0.5$. The open circles correspond to the tips of the splash jets from each liquid

In figure 2, streamlines are shown for liquid wedges with angle $\alpha' = \alpha = 10^\circ$ at two different density ratios. For the case $\rho/\rho' = 1$ in Fig. 2a the flow obviously should be symmetric respect to x-axis, and the result shows the accuracy of computation for determining the interface. For the case of density ratio $\rho/\rho' = 0.5$ in Fig.2b the interface is pushed towards the heavier liquid. The tip of the lighter liquid is shared by its free surface and interface. However the tip of the heavier liquid is further out and is on its free surface only.

The dotted lines in Fig. 2 and 3 show the free surfaces of the undisturbed wedges. The coordinates of the dotted lines at $x = 0$ shows the velocity of the liquid wedges at infinity. When V' is chosen as the reference velocity, then V is determined form the condition that the pressures at infinity for both wedges are the same.

The streamlines for the angle $\alpha' = 30^\circ$ and $\alpha = 90^\circ$ are shown in Fig 3a at $\rho/\rho' = 1$, and in Fig 3b at $\rho/\rho' = 0.5$. It is seen that for the both cases the tip is directed into the upper liquid and forms a closed cavity.



(a) (b)

Figure 3. Streamline for $\alpha' = 30°$, $\alpha = 90°$ at density ratio (a) $\rho/\rho' = 1$ and (b) $\rho/\rho' = 0.5$.

Inspection of the results shows that for the ratio $\rho/\rho' = 0.5$ the cavity slightly increased while the velocity of the lower liquid (shown by y coordindate of the dotted lines at x=0) is nearly doubled. The cavity formation and secondary impacts caused by direction of the splash jet for the case of $\rho/\rho' = 1$ were discussed in [6] and [11].

3. ASSYMPTOTIC ANALYSIS

In this section, we consider the problem of liquid wedge impact with $\rho'/\rho \gg 1$. As the first step, we consider the flow corresponding to $\rho'/\rho \to \infty$. In such a case, the liquid of ρ' is unaffected by that of ρ. It moves forward just like a rigid body. When $\rho'/\rho \gg 1$, its difference to the case of $\rho'/\rho \to \infty$ is linearized and boundary conditions are imposed on the surface corresponding to $\rho'/\rho \to \infty$.

In the asymptotic analysis, the lower liquid is considered to be at rest, while the upper liquid has velocity $\bar{V} = V + V'$. The velocity potential $\bar{V}^2 t\phi(x,y)$ in the self-similar variables x, y close to the vertex $x = 0,\ y = -1$ can be presented in the local polar coordinates r and θ , where $x = r\cos\theta,\ y = -1 + r\sin\theta$, within $\beta = \frac{\pi}{2} - \alpha' < \theta < \pi/2$, as

$$\phi(r,\theta) = \phi_0 - r\sin\theta + \sum_{n=1}^{\infty} \phi_n r^{nk} \cos\left[nk\left(\theta + \frac{\pi}{2}\right)\right], \quad k = \frac{\pi}{\pi - \alpha'} \qquad (7)$$

with undetermined coefficients ϕ_n. The second term in this representation corresponds to the undisturbed liquid wedge or at $\rho'/\rho \to \infty$. The angle β is the deadrise angle of the rigid wedge, or the angle of its surface with x axis. Note that the local solution depends on the angle α' only through the coefficients ϕ_n. The linear Bernoulli equation provides the pressure along the wedge surface, $\theta = \beta$,

$$p(r,\beta) = -\rho'\bar{V}^2 \begin{pmatrix} \phi_0 - r\sin\theta + \sum_1^\infty \phi_n (-1)^n (1-kn) r^{nk} \\ +\frac{1}{2}\left[\sum_1^\infty \phi_n kn(-1)^n r^{nk-1}\right]^2 \end{pmatrix}. \qquad (8)$$

This gives

$$\frac{\partial p}{\partial r}(r,\beta) \sim -\rho' V^2 \phi_1^2 k^2 (k-1) r^{2k-3}, \quad (r \to 0),$$

where

$$k-1 = \frac{\pi/2-\beta}{\pi/2+\beta} > 0, \qquad 2k-3 = 3\frac{\pi/6-\beta}{\pi/2+\beta}.$$

The pressure gradient may be singular at the vertex if $\beta > \pi/6$ and $\phi_1 \neq 0$. However, the pressure is finite at this point.

It is seen that the flow velocity in the heavy wedge may be singular at the vertex if the deadrise angle of this wedge is greater than 30 degrees. The exact flow behaviour from the asymptotic solution and its comparison with the solutions of Eqs.(2) and (3) will be discussed in the workshop.

The singularity of the velocity is of the same order as the singularity of the pressure gradient in the lighter liquid at the vertex. Integrating the kinematic condition on the interface between the liquids, we find that the deflection of the interface is also singular at the vertex. The order of the singularity is the same as that of the normal velocity. Therefore, an inner region should be introduced near the vertex, where the flows of both liquids are coupled.

4. CONCLUSIONS

We have presented a nonlinear solution to the self-similar problem of an impact between liquid wedges of different densities. Particular attention is given to the splash jet containing both liquids whose quantities depends on the wedge angles of the liquids and their density ratio. It is found that the interface is pushed towards the heavier liquid. The tips of the two liquids with different densities are found to be at different places.

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